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STATISTICAL FLUCTUATIONS IN THE EXCITATION FUNCTIONS FROM $C^{12}(C^{12},\alpha)Ne^{20}$

BY

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CONTENTS

Synopsis

The theory of statistical fluctuations in the cross sections for the energy region of completely overlapping levels is developed for the case of incident particles being zero spin and identical. To facilitate analysis of data, the development is in terms of the correlation coefficient, which is proportional to the difference between the mean square cross section and the square of the mean cross section. Simple expressions are derived relating the correlation coefficient with the number of degrees of freedom in the probability distribution of the cross section. For the differential cross sections, the number of degrees of freedom is twice the effective number of magnetic quantum substates; for the cross section integrated over angle, the number of degrees of freedom depends upon the number of compound spins, the size of the compound spin, and the spin of the final nucleus. Special cases are cited for which selection rules and spherical harmonics reduce the number of degrees of freedom.

Cross section data from the reaction $C^{12}(C^{12}, \alpha)Ne^{20}$ are analysed. Level densities and widths calculated from statistical theories of levels for all energies but the lowest confirm that the populated levels are completely overlapping. Satisfactory agreement is found between the number of degrees of freedom in the cross section distribution as determined from the data and from theory.

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1. Introduction

The theory of fluctuations in the excitation functions (cross section versus energy) in compound nuclear reactions has recently been developed by ERICSON [1] [2] [3] [4] and by BRINK et al. [4] [5] in the region of overlapping compound resonances. Correlation functions and distributions play important roles in fluctuation theory. On the basis of assumptions for the correlation between reduced widths for different channels, ERICSON [2] gives a theory which predicts the self- and cross-correlation functions for excitation functions in reaction channels. In [5] ERICSON calculates the distribution of the cross section in the spin zero case. Comparison with data has been made by ALMQVIST et al. [6].

In this paper, the fluctuation theory has been formulated with special reference to analysis in terms of distributions. The angular momentum effects on the cross section distributions are discussed (see also ref. [4]). It is described how one can make quantitative predictions of the cross section distributions from the calculation of absolute cross sections in the theory of statistical reactions. The fluctuation theory has been applied to analyse the strongly fluctuating cross sections in the reaction $C^{12}(C^{12},\alpha)Ne^{20}$, which has been studied experimentally by ALMQVIST et al. [6] [7] [8] and by BORGGREEN et al. [9] [10]. The reaction $C^{12}(C^{12},\alpha)Ne^{20}$ is of particular interest in another context since the resonance behaviour of the cross section has been used in the argumentation for the existence of "quasimolecular" states in nuclei [7].

2. Theory

2.1 Reaction formalism

Consider a reaction which is caused by the collision of two identical particles of spin 0 and *CM* kinetic energy ε_i (entrance channel i) where the projectile moves along the z-axis (fig. 1). The angular momentum of the total system is J, which for incident identical bosons is even. After the collision the emitted particle with spin s and spin projection μ moves in the

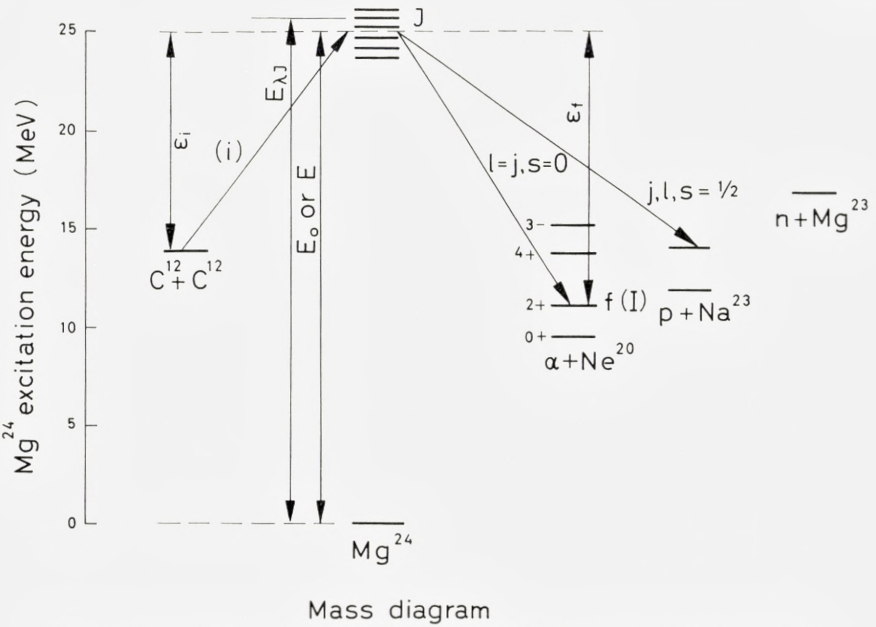

Fig. 1. Mass diagram for the reaction through the compound system Mg^{24}.

direction (θ, φ). The index f is used to identify the set $f = \{I, s\}$ of the final nucleus and particle spins. Their projections are $\{M, \mu\}$. The relative orbital angular momentum in the exit channel is l, and $\mathbf{l} + \mathbf{s} = \mathbf{j}$.

With this notation one can write the differential cross section for a reaction $i \to \{f, M, \mu\}$:

$$\left.\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{if M\mu} &= \Bigg| \sum_{\substack{\text{even } J=0 \\ |J-I| \le j \le J+I \\ l = j \pm s}}^{\infty} a(i,J,j,l,f) \times \\ (i)^l (l, s, -M-\mu, \mu \mid j-M)(jI, -MM \mid J0)\, Y_{l,-M-\mu}(\theta,\varphi)\Bigg|^2,\end{aligned}\right\} \quad (1)$$

where one has to remember conservation of parity in the sum. The vector coupling coefficients are as defined in PRESTON [11].

In an arbitrary channel c we define $\lambda\!\!\!^{-}_c = 1/k_c = \hbar/\sqrt{2m_c\varepsilon_c}$, where m_c is the reduced mass and ε_c the kinetic energy.

In the theory of resonance reactions [12] the reaction amplitude for $f \neq i$ can be written

$$a(i,J,j,l,f) = -i \cdot 2^{\frac{1}{2}} \pi^{\frac{1}{2}} \lambda\!\!\!^{-}_i (2J+1)^{\frac{1}{2}} \sum_{\lambda} \frac{g_{\lambda J i} g_{\lambda J j l f}}{E_{\lambda J} - E - \frac{i}{2}\Gamma_{\lambda J}}. \quad (2)$$

The resonance at energy $E_{\lambda J}$ has a total width $\Gamma_{\lambda J} = \sum_c \Gamma_{\lambda Jc}$, which is a sum over all open channels c of partial widths $\Gamma_{\lambda Jc} = |g_{\lambda Jc}|^2$. The channel amplitude $g_{\lambda Jc}$ is the product $2^{\frac{1}{2}} P_c \alpha_{\lambda Jc}$, where P_c is the penetration factor [11] evaluated at the channel radius R_c and is a function of l and $k_c R_c$. The reduced width $\alpha_{\lambda Jc}$ is generally a complex number which is proportional to the radial wave function evaluated at the channel radius R_c for the resonance λJ. For the present case of identical projectiles, all resonances have even parity, and the amplitude (2) has an extra factor of $\sqrt{2}$ compared to the general case.

When M and μ are not determined, the differential cross section resulting from the sum of the quantities in (1) is

$$\left(\frac{d\sigma}{d\Omega}\right)_{if} = \sum_{M,\mu} \left(\frac{d\sigma}{d\Omega}\right)_{ifM\mu}, \tag{3}$$

and the cross section integrated over all angles is, with use the of (1) and (3),

$$\sigma_{if} = \sum_{M,\mu} \int \left(\frac{d\sigma}{d\Omega}\right)_{ifM\mu} d\Omega = \sum_{\substack{\text{even } J=0 \\ |J-I| \le j \le J+I \\ l = j \pm s}}^{\infty} |a(i,J,j,l,f)|^2. \tag{4}$$

In equations (3) and (4) the cross sections are sums of incoherent quantities, and it is just this summation that reduces the fluctuations in the cross sections as discussed in the next section.

To condense the notation, the sums of incoherent quantities in (3) and (4) are rewritten simply as

$$\frac{d\sigma}{d\Omega} = \sum_{M,\mu} \left(\frac{d\sigma}{d\Omega}\right)_{\alpha}, \qquad \text{where} \qquad \alpha = \{i,f,M,\mu\}, \tag{5a}$$

$$\sigma = \sum_{J,j,l} \sigma_{\beta}, \qquad \text{where} \qquad \beta = \{i,J,j,l,f\}. \tag{5b}$$

2.2 The statistical properties of cross sections

In the theory of reactions through resonances, the conditions for purely statistical reactions are that the channel amplitudes obey the randomness assumption [2]

and
$$\left.\begin{aligned} &<g_{\lambda Jc}g_{\lambda Jc'}> = 0 \quad \text{for} \quad c \neq c' \\ &<g_{\lambda Jc}g_{\lambda' J'c'}> = 0 \quad \text{for} \quad J \neq J' \quad \text{and} \quad c \neq c', \end{aligned}\right\} \tag{6}$$

where the averages are taken over the resonances λ or $\{\lambda, \lambda'\}$ in an interval ΔE containing many resonances. In the case $\Gamma_{\lambda J} >> \langle D_J \rangle$, where $\langle D_J \rangle$ is the average energy difference between resonances with a given compound spin J and a given parity, it follows from (6) that the real and the imaginary part of the reaction amplitude $a(i, J, j, l, f)$ are statistically independent and gaussian distributed around zero with the same dispersion. Then the probability distributions of the cross sections $\left(\frac{d\sigma}{d\Omega}\right)_\alpha$ and σ_β, taken over an energy range $\Delta E >> \Gamma_{\lambda J}$, are χ^2 distributions of 2 degrees of freedom, i.e., decreasing exponentials [1] [3] [5] (see Appendix).

To facilitate the analysis of data in terms of the number of degrees of freedom in the statistical fluctuations, it is convenient to consider correlation functions [1], which involve the first and second moments of the cross section distributions. A χ^2 distribution of 2 degrees of freedom has the property that the correlation coefficient of the variable σ_β is

$$R = \frac{<(\sigma_\beta - <\sigma_\beta>)^2>}{<\sigma_\beta>^2} = \frac{<\sigma_\beta{}^2>}{<\sigma_\beta>^2} - 1 = 1. \tag{7}$$

Since a cross section which is fluctuating with a χ^2 distribution of 2 degrees of freedom plays a fundamental role for compound reactions in the region of overlapping levels, we shall call it a *basic cross section*.

The integrated cross section $\sigma = \sum_\beta \sigma_\beta$ fluctuates as a result of the fluctuations in the various contributing σ_β, but the amount of the fluctuation of course depends on the relations between the various σ_β. These cross section relations will now be considered in terms of the relations between the reaction amplitudes $a(i, J, j, l, f)$. The randomness assumptions (6) essentially state that the average of the product of two different amplitudes over an interval $\Delta E >> \Gamma_{\lambda J}$ is zero:

$$<a(i, J, j, l, f)\, a(i, J', j', l', f')> = 0 \tag{8}$$

[1] The *cross correlation function* of two functions $\sigma_1(E)$ and $\sigma_2(E)$ as a function of the incremental energy δ is

$$R_{1,2}(\delta) = \frac{<[\sigma_1(E) - <\sigma_1(E)>][\sigma_2(E+\delta) - <\sigma_2(E)>]>}{<\sigma_1(E)><\sigma_2(E)>}.$$

The *self-correlation function* for a function $\sigma_1(E)$ is $R_{1,1}(\delta)$. The *correlation coefficient* R is defined as the self-correlation function for $\delta = 0$.

for $\{J,j,l,f\} \neq \{J',j',l',f'\}$. The equation (7), which characterizes the probability distribution in the basic cross section, states that

$$\frac{<|a(i,J,j,l,f)|^4>}{<|a(i,J,j,l,f)|^2>^2} = 2. \tag{9}$$

We wish to analyse actual cross sections σ in terms of the correlation coefficient

$$R = \frac{<\sigma^2>}{<\sigma>^2} - 1. \tag{10}$$

The relation (9) can be used to reduce (10) when it is expanded in the basic cross sections of σ. Also in the expansion are the averages $<|a(i,J,j,l,f)|^2|a(i,J',j',l',f')|^2>$, which to simplify involve more than the randomness assumptions. If one assumes, however, the *reaction amplitudes* $a(i,J',j,l,f)$ and $a(i,J',j',l',f')$ to be completely *independent* for $\{i,J,j,l,f\} \neq \{i,J',j',l',f'\}$ in the interval ΔE, the simplification of the independence approximation

$$<|a(i,J,j,l,f)|^2|a(i,J',j',l',f')|^2> = <|a(i,J,j,l,f)|^2><|a(i,J',j',l',f')|^2> \tag{11}$$

follows. In condensed notation for $\beta = \{i,J,j,l,f\}$ and $\beta' = \{i,J',j',l',f'\}$ where $\beta \neq \beta'$,

$$<\sigma_\beta \sigma_{\beta'}> = <\sigma_\beta><\sigma_{\beta'}>. \tag{12}$$

Similar expressions follow for all higher powers of the amplitudes $a(i,J,j,l,f)$ as consequence a of the independence assumption. It is, however, apparent that the use of R in the analysis tests the independence of the basic cross sections only to the order involved in (12), but not the full independence of reaction amplitudes a.

In the differential cross section (3), the cross sections $\left(\frac{d\sigma}{d\Omega}\right)_{ifM\mu}$ characterized by the individual magnetic quantum numbers are all functions of the same set of energy-dependent amplitudes $a(i,J,j,l,f)$ with coefficients which are products of the $\{M,\mu\}$-dependent, but energy-independent, vector coupling coefficients and spherical harmonics. Since these energy-independent coefficients vary in sign from one set α to another, one obtains to the extent of

$$<\left(\frac{d\sigma}{d\Omega}\right)_\alpha \left(\frac{d\sigma}{d\Omega}\right)_{\alpha'}> \approx <\left(\frac{d\sigma}{d\Omega}\right)_\alpha><\left(\frac{d\sigma}{d\Omega}\right)_{\alpha'}>, \tag{13}$$

where $\alpha = \{i,f,M,\mu\}$ and $\alpha' = \{i,f',M',\mu'\}$, an approximate independence between the various basic cross sections $\left(\frac{d\sigma}{d\Omega}\right)_{ifM\mu}$ contributing to the differential cross section. However, the number N_1 of such approximately independent basic cross sections cannot exceed the number N_2 of different contributing amplitudes $a(i,J,j,l,f)$ to the cross section. When N_1 is small compared to N_2, which is the case in all cross sections examined in the reaction considered, one expects (13) to be valid with sufficient accuracy. We shall therefore in all cases examined consider the basic cross sections in (5a) and (5b) to be independent to the extent of (12) or (13).

With the use of (12) and (5b), the correlation coefficient (10) of integrated cross sections can be reduced to

$$R = \frac{\sum\limits_{J,j,l} <\sigma_\beta>^2}{<\sum\limits_{J,j,l} \sigma_\beta>^2} \equiv \frac{2}{\nu_{\text{eff}}}. \tag{14}$$

For the differential cross sections (5a), one can similarly define a correlation coefficient which with the use of (13) becomes

$$R = \frac{<\left(\frac{d\sigma}{d\Omega}\right)^2>}{<\frac{d\sigma}{d\Omega}>^2} - 1 \simeq \frac{\sum\limits_{M,\mu} <\left(\frac{d\sigma}{d\Omega}\right)_\alpha>^2}{<\sum\limits_{M,\mu}\left(\frac{d\sigma}{d\Omega}\right)_\alpha>^2} \equiv \frac{2}{\nu_{\text{eff}}}. \tag{15}$$

In analogy with the formal ν of (A3) in the Appendix, (14) and (15) define ν_{eff}, which characterizes the fluctuation and which is generally not an integer.

If all the averages of n independent basic cross sections are equal in (14) or (15), the correlation coefficient is simply

$$R = \frac{n}{n^2} = \frac{1}{n}. \tag{16}$$

Thus, the correlation coefficient of the cross section directly reflects the number of independent basic cross sections in (5a) and (5b). The result of the simple theory is then $\nu_{\text{eff}} = 2n$, where ν_{eff} becomes an integer, and the distribution function of the cross section is a χ^2 distribution of $2n$ degrees of freedom.

When ν_{eff} is a non-integer, we expect that the cross section distribution is similar to a χ^2 distribution which, however, is only defined for integer values of ν. As seen in the Appendix, it is possible to use ν as a continuous variable, and therefore one expects the cross section distribution in this case to be approximately (A1) with $\nu = \nu_{\text{eff}}$ (a gamma distribution).

2.3 Special cases for the differential cross section

Selection rules and the spherical harmonic properties can result in some basic cross sections in (5a) being zero. This fact strongly affects the fluctuation, and therefore we analyse these cases in some detail.

For alpha-particle channels, $s = 0$, $j = l$, and $\mu = 0$. The following selection rules affect the number of $\alpha = M$ values.

a) Because $(lI - MM|J0)Y_{l-M} = (-)^{I+l-J}(lIM - M|J0)Y^*_{lM}(-)^M$, we see that $\dfrac{d\sigma}{d\Omega} = \sum_M \left(\dfrac{d\sigma}{d\Omega}\right)_M = \sum_{M \geq 0} \left(\dfrac{d\sigma}{d\Omega}\right)_M (2 - \delta_{M0})$. This decreases the number of M values to the value of $I+1$ or $l+1$, whichever is smaller.

b) The vector coupling coefficient is $(lI00|J0) = 0$ for odd values of $l+I+J$; this eliminates the $M = 0$ values for the unnatural parity exit channels. (This rule is also valid for incident spin-zero nonidentical particles).

c) For $\theta = 0$ or π, $Y_{lM}(\theta, \varphi) = 0$ for $M \neq 0$. In these cases only the one $M = 0$ value contributes to the cross section.

d) For $\theta = \dfrac{\pi}{2}$, the property $P_{lM}(\cos\theta) = (-)^{l+M}P_{lM}(-\cos\theta)$ of the associated Legendre polynomials gives even M for even parity exit channels and odd M for odd parity exit channels.

Some results of a) through d) are presented in table I. The neutron and proton exit channels can be analysed in a similar way, but these are not required in the present analysis. If the magnitude of $<\left(\dfrac{d\sigma}{d\Omega}\right)_\alpha>$ over the energy interval considered is the same for all α values, the simple theory (16) gives ν_{eff} equal to twice the number of M values allowed by a) through d) above.

TABLE I. Number of nonvanishing independent basic cross sections (effective number of α values) in the differential cross section for the reaction $C^{12}(C^{12},\alpha)Ne^{20}$. (See sections 2.2 and 2.3.)

Spin I and parity of the final state in Ne^{20}	Effective number of α values		
	At an arbitrary angle θ	$\theta = 0°$	$\theta = 90°$
0+	1	1	1
0−	0	0	0
1+	2	0	1
1−	1	1	1
2+	3	1	2
2−	2	0	1
3+	4	0	2
3−	3	1	2
4+	5	1	3
4−	4	0	2
5+	6	0	3
5−	5	1	3
6+	7	1	4
6−	6	0	3
7+	8	0	4
7−	7	1	4

2.4 Average cross sections

To predict ν_{eff} from (14) or (15), it is necessary to know the average cross sections $<\sigma_{if}>$ or $<\left(\frac{d\sigma}{d\Omega}\right)_{if}>$, respectively. With the randomness assumptions (6) for reaction amplitudes, the average of the differential cross section is

$$\left.\begin{aligned} <\left(\frac{d\sigma}{d\Omega}\right)_{if}> &= \sum_{M,\mu} <\left(\frac{d\sigma}{d\Omega}\right)_{\alpha}> \\ &= \sum_{M,\mu} <\Big| \sum_{J,l,j} a(i,J,j,l,f)(l,s,-M-\mu,\mu|j,-M) \\ &\qquad \times (jI,-MM|J0)\,Y_{l,-M-\mu}(\theta,\varphi)|^2> \\ &= \sum_{\substack{J,l,j \\ M,\mu}} <|a(i,J,j,l,f)|^2>(l,s,-M-\mu,\mu|j,-M)^2 \\ &\qquad \times (jI,-MM|J0)^2 Y_{l,-M-\mu}(\theta,\varphi)|^2. \end{aligned}\right\} \quad (17)$$

Similarly, the average of the integrated cross section is

$$<\sigma_{if}> = \sum_{J,j,l} <\sigma_\beta> = \sum_{J,j,l} <|a(i,J,j,l,f)|^2>. \tag{18}$$

Both (17) and (18) contain the average $<|a(i,J,j,l,f)|^2>$ which, with the use of (2), can be written

$$<|a(i,J,j,l,f)|^2> = 2\pi\lambda_i^2(2J+1)<\left|\sum_\lambda \frac{g_{\lambda Ji}g_{\lambda Jjlf}}{E_{\lambda J} - E - \frac{i}{2}\Gamma_{\lambda J}}\right|^2>. \tag{19}$$

With the use of (5), (19) is reduced to

$$<|a(i,J,j,l,f)|^2> = 2\pi\lambda_i^2(2J+1)\frac{2\pi}{<D_J>}<\frac{\Gamma_{\lambda Ji}\Gamma_{\lambda Jjlf}}{\Gamma_{\lambda J}}>, \tag{20}$$

where $<D_J>$ is the average spacing between resonances with a definite spin J and parity. The cross section average is taken over an energy interval $\Delta E >> \Gamma_{\lambda J}$ around E_0, and the relative error (a finite sample effect [2]) is of the order of $(\Gamma_{\lambda J}/\Delta E)^{\frac{1}{2}}$. The total width $\Gamma_{\lambda J}$ is nearly constant between nearby states λ of given J.

We assume $\Gamma_{\lambda Ji}$ and $\Gamma_{\lambda Jjlf}$ to be statistically independent and identify $2\pi\frac{<\Gamma_{\lambda Jc}>}{<D_J>}$ with the transmission coefficient T_c^J calculated from the optical model for a particle with *CM* energy ε_c in channel $\{c,J\}$. Then (20) can be transformed to the usual Hauser-Feshbach expression [13]

$$<|a(i,J,j,l,f)|^2> = 2\pi\lambda_i^2(2J+1)T_i^J(\varepsilon_i)\frac{T_f^J(\varepsilon_f)}{\sum T_c^J(\varepsilon_c)}, \tag{21}$$

and the total width is

$$<\Gamma_{\lambda J}> \equiv \Gamma_J = \frac{<D_J>}{2\pi}\sum_c T_c^J(\varepsilon_c). \tag{22}$$

The relations (21) and (22) are justified only for $T_c^J << 1$, but here $T_c^J \sim 1$ for most of the channels. MOLDAUER [14] [15] and KRIEGER and PORTER [16] have considered this problem in some detail and have developed expressions for the average cross section when $T_c^J \sim 1$. These expressions require a detailed knowledge about the statistical properties of the compound system, and so we shall simply use (21) for the average cross section. The transmission coefficients in (21) depend on the kinetic energy ε_c in channel c. They are determined as

$$T_c = 1 - \exp(-4Im\delta_c), \tag{23}$$

where δ_c is the phase shift in channel c determined from optical model calculations. We assume that all the parameters of the optical potentials are the same for the elastic scattering on a nucleus in an excited state as for the elastic scattering on a nucleus in the ground state. We therefore omit the labels J and f on the transmission coefficient which is then labelled by the quantum number l for alpha particles, and by j, l for neutrons and protons.

3. Comparison with experiment

3.1 General assumptions

One of the striking features of the $C^{12}(C^{12}, \alpha)Ne^{20}$ reaction is the strongly fluctuating excitation functions [6] [7] [8] [9] [10].

In fig. 2 is shown the cross correlation function $R_{1,2}$ between the integrated cross sections [6] [8] of the ground and first excited state in the residual nucleus Ne^{20}. This lack of correlation between the observed peaks in the cross section for different exit channels indicates that the reaction proceeds via a compound mechanism in the region of overlapping resonances. Whether there is also a direct amplitude cannot be seen in a forward-backward asymmetry. This is because the colliding particles are identical, and thus the angular distribution is automatically symmetric around $\theta = 90°$. The statistical model has earlier been applied to analyse three cross section peaks observed in the integrated cross section to the ground state channel under the assumption that they are isolated resonances [8]. We shall not use this last assumption, but instead analyse the reaction as a pure compound reaction with overlapping resonances and then look for possible discrepancies between the theoretical calculations and the experiments.

3.2 Average cross sections and fluctuation widths

In the optical model calculations the following potential was used [17]:

$$(\mathrm{ReV} + i\,\mathrm{ImV})\varrho(r) + V_{so}\left(\frac{\hbar}{m_\mu c}\right)^2 \frac{1}{r}\frac{d\varrho(r)}{dr}\,\sigma\cdot\mathbf{l} + V_{coul},$$

where

$$\varrho(r) = 1/\left(\exp\frac{r-R}{a} + 1\right)$$

and

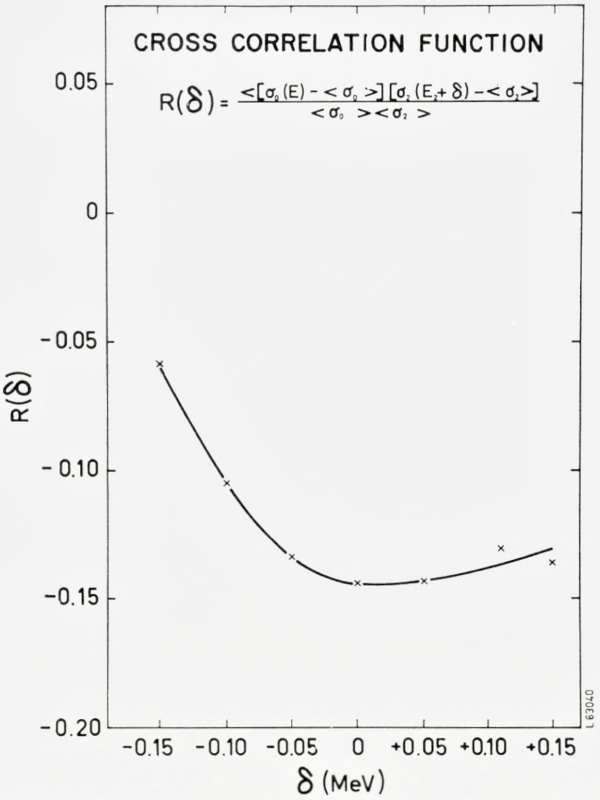

Fig. 2. Cross correlation function for the observed integrated cross sections in the $\alpha + Ne^{20}_{+0}$ and $\alpha + Ne^{20}_{2+}$ channels. Data are from ALMQVIST et al.[6] [8].

$$V_{\text{coul}} = \begin{cases} \dfrac{Z_1 Z_2 e^2}{r} & \text{for} \quad r \geqslant R \\ \dfrac{Z_1 Z_2 e^2}{2R}\left(3 - \dfrac{r^2}{R^2}\right) & \text{for} \quad r < R \end{cases}$$

with the parameters given in table II. This calculation gives transmission coefficients $T_{lj}(\varepsilon_f)$, which are then used in the Hauser-Feshbach expression (21). The calculation T_l for the important channels of alpha particles and C^{12} are shown in fig. 3. (For the T_{lj} of the neutron and proton channels, see BORGGREEN et al.[9], where the j dependence was neglected).

To calculate $\sum_c T_c$, it is in principle necessary to know all the spins,

TABLE II. Optical model parameters used for $C^{12}+C^{12}$, $\alpha+Ne^{20}$, $p+Na^{23}$, and $n+Mg^{23}$ in section 3.2.

Channel	ReV (MeV)	ImV (MeV)	a (fm)	R (fm)	Reference
$C^{12}+C^{12}$	48	5.75	0.575	5.838	18)
$\alpha+Ne^{20}$	50	5.3	0,576	4.95	19)
$p+Na^{23}$	40	9	0.73	3.695	20)
$n+Mg^{23}$	45.5	9.5	0.65	3.553	21)

energies, and parities of the states in the residual nuclei. For the lower energies of the compound system there are so few exit channels that one has this knowledge, but at higher excitation energies one has to take advantage of both the experimental and theoretical knowledge of the density of states and spin distributions. This results in an integral over unidentified states in addition to the sum over identified states. For the higher excitation energy region of unidentified states a theoretical expression [22] with adjustable parameters for the level density including states with both parities and all spins

$$\omega(E^*) = C_\omega \frac{\exp\left[2\sqrt{a_\omega U}\right]}{(U+t)^2} \tag{24}$$

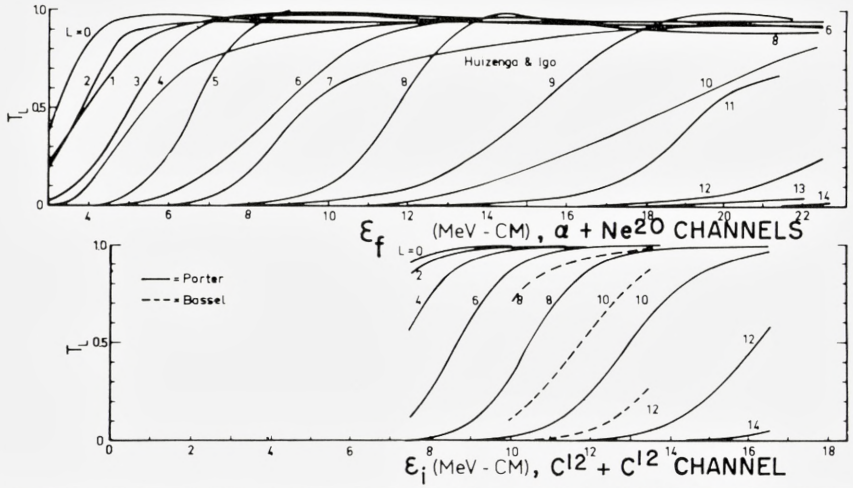

Fig. 3. Transmission coefficients from optical model calculations of $C^{12}+C^{12}$ and $\alpha+Ne^{20}$ channels. The optical model results from the parameters of BASSEL and DRISKO [R. H. Bassel and R.M. Drisko, *Proceedings of the International Conference on Nuclear Structure*, Kingston (University of Toronto Press, Toronto, 1960), p. 212] were not used in the present calculations.

TABLE III. Parameters used in the level distributions (24) for Ne^{20}, Mg^{23}, Na^{23}, and Mg^{24}.

Nucleus	C_ω(MeV)	a_ω(MeV^{-1})	Δ(MeV)	t(MeV)
Ne^{20}	0.539	2.240	4.95	2.75
Mg^{23}, Na^{23}	1.94	2.035	2.48	2.22
Mg^{24}	0.217	2.99	4.58	2.11

TABLE IV. "Spin cutoff factors" for Ne^{20}, Mg^{23}, Na^{23}, and Mg^{24} in (25).

Nucleus	S^2
Ne^{20}	5.55
Na^{23}, Mg^{23}	6.53
Mg^{24}	6.84

was used, where $U = E^* - \Delta$, E^* is the excitation energy, and Δ is the pairing energy. Pairing energy values of CAMERON [23] were used. The nuclear temperature t and the constants C_ω and a_ω were determined from best fits to known level schemes [24] [25]. The constants used in the calculation are shown in table III.

The spin distribution was chosen to be the usual expression

$$G(I) = C_G(2I+1)\exp\left[-\frac{I(I+1)}{2S^2}\right], \tag{25}$$

where C_G is the normalization constant. The "spin cutoff factor" S was considered to be independent of the excitation energy E^*. Values were obtained from best fits of the functional relation $S^2 \propto A^{7/6}$ to the values [26] given for the three nuclei Al^{26}, S^{33}, and A^{37}. The "spin cutoff factors" S used are given in table IV.

The part of $\sum_c T_c$ that was an integration over unidentified levels was calculated with the parameters in tables III and IV for alpha-particle, carbon, proton, and neutron exit channels[1]. In this region of high excitation energy, the average distance between levels of a definite spin and parity for the residual nucleus is

$$<D_I> = \frac{2}{\omega(E^*)G(I)}. \tag{26}$$

[1] Optical model calculations were kindly made by L. Stewart at Los Alamos Scientific Laboratory and E. Auerbach at Brookhaven National Laboratory.

TABLE V. Calculated values of $<D_J>$, $\sum_c T_c$, and $\Gamma_J = \frac{<D_J>}{2\pi}\sum_c T_c$ for Mg^{24} as a function of the excitation energy E_0 in Mg^{24}. The contribution to $\sum_c T_c$ from the neutron and C^{12} channels is small. The contribution from the proton channels is largest for $J = 0, 2, 4$, the alpha-particle and proton channels contribute nearly equally for $J = 6$, and the alpha-particle channel contributions are largest for $J = 8, 10$.

J	ε_i (MeV)	E_0 (MeV)	$<D_J>$ (keV)	$\sum_c T_c$	Γ_J (keV)
0	8	21.9	28.7	37	171
	12	25.9	8.5	152	206
	16	29.9	2.8	560	249
2	8	21.9	8.9	101	143
	12	25.9	2.6	487	202
	16	29.9	0.9	1953	280
4	8	21.9	13.7	44	157
	12	25.9	4.1	381	249
	16	29.9	1.3	1571	325
6	8	21.9	47.4	30	227
	12	25.9	14.1	169	379
	16	29.9	4.6	705	516
8	8	21.9	318	8.0	403
	12	25.9	94.7	47.5	717
	16	29.9	30.9	204.0	1003
10	8	21.9	4160	1.2	830
	12	25.9	1220	11.1	2150
	16	29.9	396	47.4	2910

(A corresponding equation gives the spacing $\langle D_J \rangle$ for the compound nucleus). The factor 2 in the numerator comes from the choice of a definite parity. In the region of low excitation energy, compilations of identified levels [24] [25] were used.

Table V shows the calculated $<D_J>$, $\sum_c T_c$, and Γ_J for the compound nucleus Mg^{24} as a function of ε_i. The calculated Γ_J can now be compared with the measured fluctuation width. (See section 4).

In figure 4 the integrated cross section calculated from (18) and (21) is shown for the ground-state channel. This cross section is a sum of contributions from only even values of the compound spins $J = l = 0, 2, 4, 6, 8, 10$ etc. Consequently, in the excitation functions rather broad intervals of the

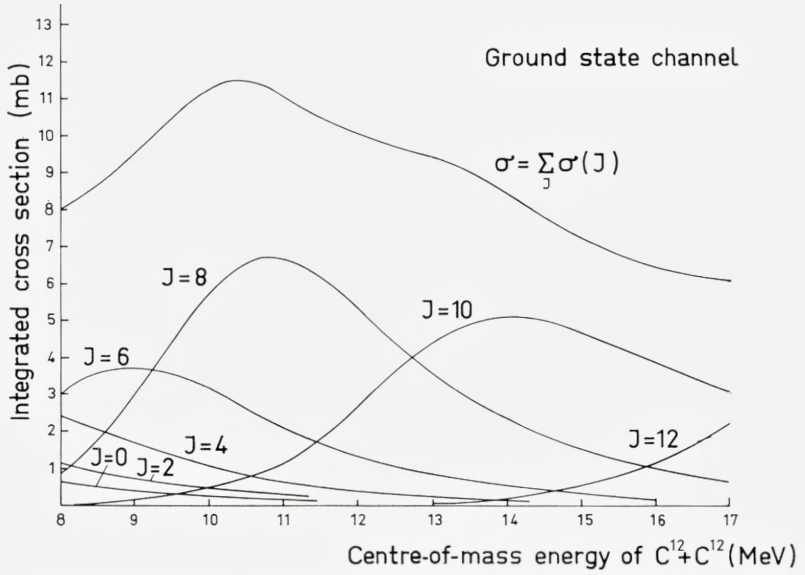

Fig. 4. Calculated integrated average cross section in the $\alpha + Ne^{20}_{0+}$ channel as a function of incident $C^{12} + C^{12}$ energy ε_i.

energy ε_i occur in which usually a single compound spin dominates in the angular distributions of this differential cross section

$$\frac{d\sigma}{d\Omega} = \left| \sum_J a(i,J,j = J,l = J,f) Y_{J,0}(\theta,\omega) \right|^2 \tag{27}$$

simplified from (1) for the $I = 0$ ground-state channel. Such angular distributions at different bombarding energies have been measured by LASSEN and OLSEN [28] in the compound spin $J = 4$ to $J = 8$ region, by BORGGREEN et al. [10] in the $J = 6$ to $J = 10$ region, and by ALMQVIST et al. [6] [8] in the $J = 8$ region. The compound spins that usually dominate are given in table VI together with their calculated and observed energy intervals.

The average cross sections calculated from (17) and (18) with the use of (21) vary slowly with energy E_0. To compare these calculated cross sections with the experimental cross sections, (17) and (18) have been averaged over energy intervals corresponding to the experimental intervals to give $\bar{\sigma}$ and $\overline{\left(\frac{d\sigma}{d\Omega}\right)}$. (See eq. (30) of the next section). The results of this calculation are shown for some examples in table VII.

TABLE VI. Intervals of the usually dominant spin J in the $C^{12}(C^{12},\alpha)Ne^{20}$ ground-state reaction.

Usually dominant spin J	Experimental ε_i energy interval (MeV)	Theoretical ε_i energy interval (MeV)
4	—10.6	— 7.9
6	—11.1	7.9— 9.4
8	9.5—15.0	9.4—12.7
10	13.0—	12.7—17.4
12		17.4—

TABLE VII. Theoretical and observed average cross sections and fluctuation parameters. "Simple theory" results are twice the values in table I. "Refined theory" refers to (29a) and (29b) of section 3.3.

E^* Ne^{20} (MeV)	Ne^{20} final state spin I and parity	Cross section	θ	Energy interval $\Delta\varepsilon_i$(MeV)	Experimental average cross section	Calculated average cross section	ν_{exp}	ν_{theor}	
								Simple theory	Refined theory
0	0+	Differential	0°	10.1—12.8	24.4 mb/sr	13.2 mb/sr	3.9	2	2.0
		Differential	90°	9.0—15.9	0.90 mb/sr	0.94 mb/sr	1.7	2	1.9
		Integrated		10.1—12.8	20 mb	10.5 mb	3.9		5.2
1.63	2+	Differential	0°	10.1—12.8	26.4 mb/sr	11.1 mb/sr	5.6	2	2.0
		Differential	90°	9.0—15.9	2.13 mb/sr	1.74 mb/sr	2.6	4	3.6
		Integrated		10.1—12.8	64 mb	28.0 mb	10.6		14.6
4.25	4+	Differential	0°	10.1—12.8	24.8 mb/sr	7.2 mb/sr	4.2	2	1.9
		Differential	90°	9.0—15.9	2.42 mb/sr	2.24 mb/sr	6.6	6	5.3
5.63	3−	Differential	0°	10.1—12.8	40.9 mb/sr	6.5 mb/sr	8.5	2	2.0
		Differential	90°	9.0—15.9	2.21 mb/sr	1.02 mb/sr	4.2	4	3.5

3.3 Fluctuation of cross sections

There are various ways of analysing the fluctuation of a function. However, in general, the information one can extract depends on prior knowledge or an assumption of the variation of the average function about which the fluctuation occurs. Therefore, the theoretical considerations can influence a fluctuation analysis of the experimental data in a somewhat arbitrary manner, although the better a possible average variation is known,

the more one can rely on the fluctuation analysis. A specific example of this is the general requirement for normalizing (to a constant cross section) the smoothed cross section, thus giving a normalized cross section including fluctuations over the energy region of the excitation function considered either experimentally or theoretically. This is to avoid giving the larger of the smoothed cross sections in the energy interval undue weight in the fluctuation analysis. However, the averaged (smoothed) cross sections calculated in section 3.2 and given in fig. 4 are already nearly constant in the energy intervals analysed, and so these normalizations are neglected in the following fluctuation analyses of the experimental data.

We now make fluctuation analyses of probability distributions shown in figs. 5–7 for the measured [6] [8] [9] [10] cross sections. Histograms of the cross section probabilities are constructed directly from the experimental points in the excitation functions, which were measured with equal intervals of energy. For these analyses, use is made of the correlation coefficients of the experimental data

$$R_{\mathrm{exp}} = \frac{\left[\left(\frac{d\sigma}{d\Omega}\right)^2_{\mathrm{exp}}\right]_{av}}{\left[\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{exp}}\right]^2_{av}} - 1 \equiv \frac{2}{\nu_{\mathrm{exp}}} \tag{28a}$$

$$R_{\mathrm{exp}} = \frac{(\sigma^2_{\mathrm{exp}})_{av}}{(\sigma_{\mathrm{exp}})^2_{av}} - 1 \equiv \frac{2}{\nu_{\mathrm{exp}}}, \tag{28b}$$

which define the experimental fluctuation parameter ν_{exp} analogous to ν_{eff} of (15) and (14), respectively. The averages are over the experimental cross sections occurring through the energy interval ΔE_0 considered. The resulting ν_{exp} values are given in figs. 5–7 and in table VII.

To provide a comparison of ν_{exp} with theory, we define the equivalent quantity ν_{theor} by the correlation coefficients

$$R_{\mathrm{theor}} = \frac{\overline{\left(\frac{d\sigma}{d\Omega}\right)^2}}{\overline{\left(\frac{d\sigma}{d\Omega}\right)}^2} - 1 \equiv \frac{2}{\nu_{\mathrm{theor}}} \tag{29a}$$

$$R_{\mathrm{theor}} = \frac{\overline{\sigma^2}}{\bar{\sigma}^2} - 1 \equiv \frac{2}{\nu_{\mathrm{theor}}}. \tag{29b}$$

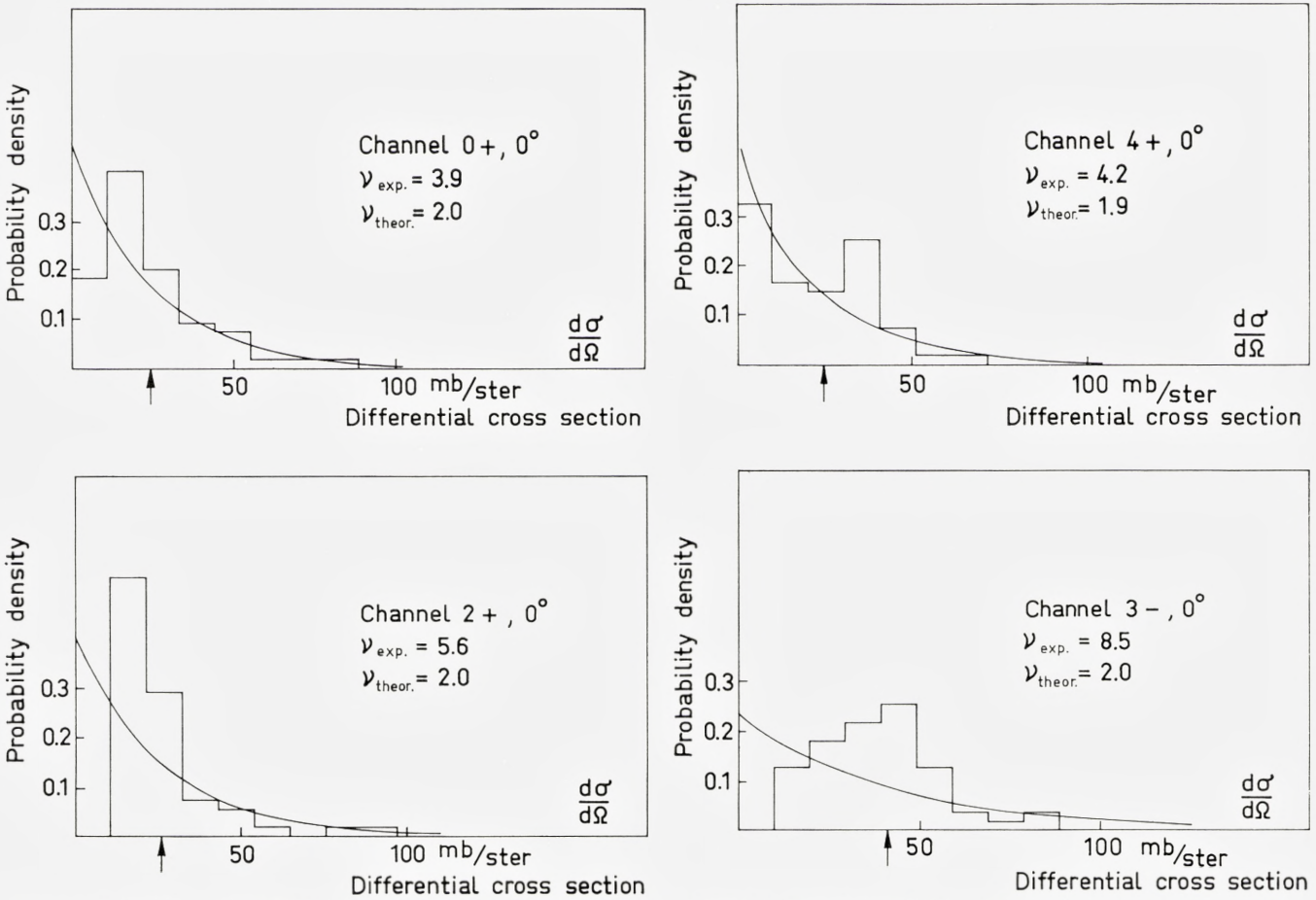

Fig. 5. Probability distributions of differential cross sections for 0° channels. Experimental data are from BORGGREEN et al.[10]. The theoretical distribution functions are normalized to the same average, which is indicated by the arrows, and area as the experimental histograms. The energy intervals are given in table VII.

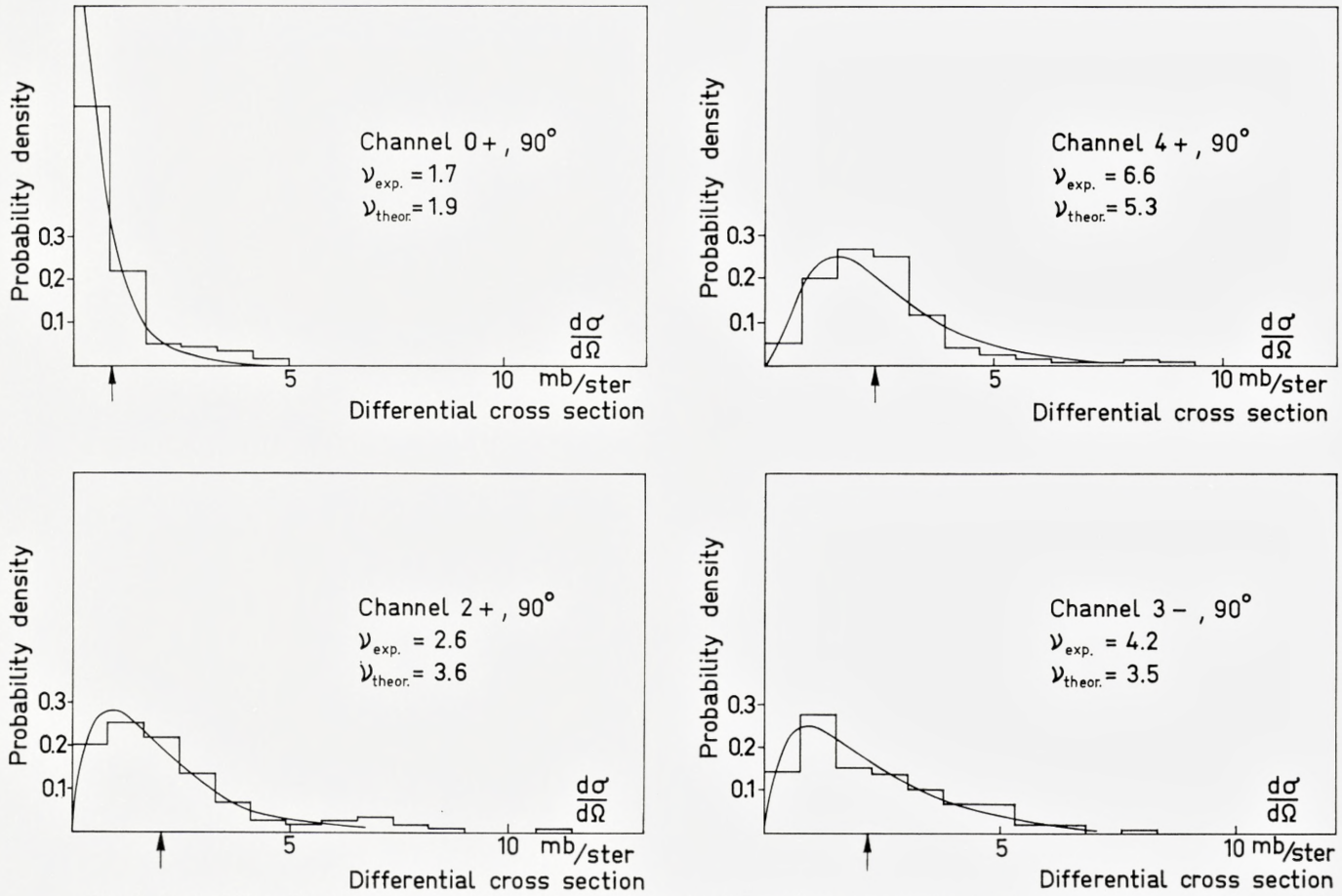

Fig. 6. Probability distributions of differential cross sections for 90° channels. See caption to fig. 5 for further details.

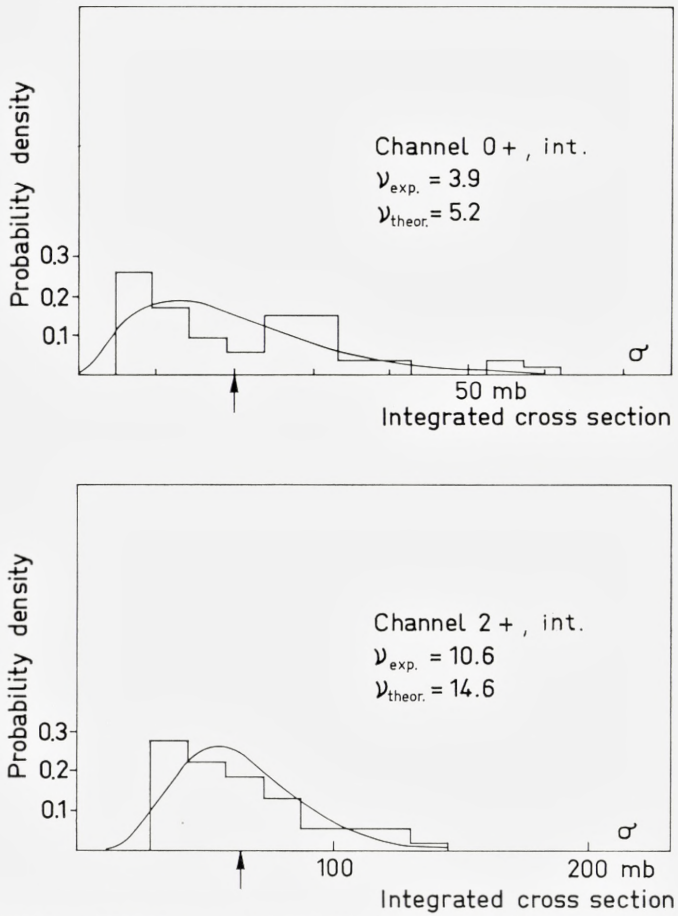

Fig. 7. Probability distributions of integrated cross sections. Experimental data are from ALMQVIST et al. [6] [8]. See caption to fig. 5 for further details.

Again, ν_{theor} is analogous to ν_{eff} of (15) and (14), respectively, in the sense that constant $<\frac{d\sigma}{d\Omega}>$ and $<\left(\frac{d\sigma}{d\Omega}\right)^2>$, or $<\sigma>$ and $<\sigma^2>$, over the energy interval ΔE_0 result in $\nu_{\text{theor}} = \nu_{\text{eff}}$.

To be comparable to the averages av in (28a) and (28b), the averages designated by bars in (29a) and (29b) must be over the same energy intervals ΔE_0. Knowing from section 2.4 that $<>$ signifies an average in the sense of the Hauser-Feshbach combination (21) of overlapping compound resonances λ, we have defined the wider interval average designated by a bar as

$$\overline{F} = \frac{1}{\Delta E_0} \int_{\Delta E_0} <F> dE_0 \qquad (30)$$

of such a <> average of a general function F over the energy interval ΔE_0. The denominator in (29a) is obtained from (17) by the square of the bar-averaged (30) sum $\sum_{M,\mu}$ of the basic cross sections $<\left(\frac{d\sigma}{d\Omega}\right)_\alpha>$, for which the reaction amplitudes a are calculated from the Hauser-Feshbach expression (21). Since $<\left(\frac{d\sigma}{d\Omega}\right)_\alpha^2> = 2<\left(\frac{d\sigma}{d\Omega}\right)_\alpha>^2$ for basic cross sections, $\overline{\left(\frac{d\sigma}{d\Omega}\right)^2}$ is similarly obtained, but from the bar-averaged sum of the squares in contrast to the square of the bar-averaged sum needed for $\left(\overline{\frac{d\sigma}{d\Omega}}\right)^2$. Similar remarks apply for the integrated cross sections, except that (18) is used with the sum $\sum_{J,j,l}$.

Calculations from (29a) and (29b) gave the ν_{theor} results shown in table VII as "refined theory" and in figs. 5–7. The numbers in the column "simple theory" in table VII are simply twice the effective number of M values from table I.

Shown together with the experimental histograms in figs. 5–7 are the χ^2 (actually gamma) distribution functions corresponding to ν_{theor} determined from (29a) and (29b). These theoretical distributions were normalized to the same average value and area as the histograms. Even if comparisons were to be made between these full distribution functions of experiment and theory, it should be emphasized that such comparisons based on the present use of the correlation coefficient would be limited in validity to the first and second moments of the cross section distributions.

3.4 Probability of dominant spin

In the ground-state channel there is a possibility of a simple, but unfortunately limited, further test of the independence approximation (13). This is the probability of an exception to the usually dominant compound spin in the energy interval of calculated dominance. We consider a mixture of two different spin values, say $J = x$ and $J = y$, having the partial average cross sections $<\sigma_x>$ and $<\sigma_y>$ with the two cross sections being independent

and having probability distributions that are decreasing exponentials. It is then easy to show that the probability of σ_x being greater than σ_y is

$$P(\sigma_x > \sigma_y) = \frac{<\sigma_x>}{<\sigma_x> + <\sigma_y>}. \tag{31}$$

In the energy interval of dominant compound spin 8, the number of angular distributions (taken with only roughly equidistant energy intervals) which show a $J \neq 8$ exception to the dominant spin was found [10] to be 1 out of 6 angular distributions, or 17%. The calculations show $<\sigma_{J=6,10}>$ to be about $<\sigma_{J=8}>/3$, which gives $P(\sigma_{J\neq 8}>\sigma_8)_{\text{theor}} = \frac{1/3}{(1/3)+1} = 25\%$, which within the poor statistics agrees with the experimental value of 17%. Approximately the same result is found [10] for the compound spin $J = 10$ region.

The very small number of observations at this time available puts severe limitations on the conclusions of this section.

4. Discussion and conclusion

The calculations result in three types of quantities which can be compared with experiment: resonance widths, absolute cross sections, and fluctuation parameters.

The calculated resonance widths shown in table V are typically 150 to 500 keV for compound spin $J = 0$ to $J = 6$ and 400 to 3000 keV for $J = 8$ and $J = 10$. These widths vary relatively little with the excitation energy of the compound system and are to be compared with the observed widths of 150 to 200 keV. Since $\sum_c T_c >> 2\pi$, one sees that the reaction is clearly in a region of overlapping levels, which is a main condition for the whole theory to be valid.

The calculated widths have uncertainties due to two effects that we now discuss. As seen from (25), the density of states of spin considerably larger than $\sqrt{2S^2}$ is very sensitive to the value of S^2. There is evidence that S^2 is not energy-independent, but rather $S^2 \propto t\Im$, where t is the nuclear temperature and $\Im$ the moment of inertia. Both t and $\Im$ increase with the excitation energy. This causes $<D_J>$ for the high spins to decrease at the higher excitation energies in the compound nucleus and, thus, a decrease in the calculated Γ_J. This is because $\sum_c T_c$ is largely originating from low lying and

low spin states in the residual nuclei for which a corresponding variation in S^2 is unimportant. On the other hand, corrections for $T_c \sim 1$ (see section 2.4) generally increase the calculated $\sum_c T_c$ and thus increase the calculated Γ_J.

The S effect tends to narrow the calculated width Γ_J for high J, but probably not as much as the discrepancy, while the correction for $T_c \sim 1$ tends to widen the calculated width. We conclude that some additional effects, such as the use of too many exit channels, account for the discrepancy between measured and calculated widths.

The calculated average cross sections are in general somewhat uncertain as a result of the rather big uncertainty in the density of final states extrapolated into the region of unidentified levels. The calculated average differential cross sections (table VII) are somewhat smaller than the observed ones for $\theta = 90°$. For $\theta = 0°$ the calculated differential cross sections are about two to six times smaller than the observed ones. This large observed 0° yield might correspond to some deviation from the randomness assumptions (6) for reaction amplitudes. In the two cases examined, the calculated integrated cross sections are about half the observed value.

The calculated fluctuation parameters ν_{theor} are generally in agreement with the experimental ones. It is seen in table VII that the "refined theory" (29a) and (29b) does not give results which differ much from the "simple theory" for the degrees of freedom in the distribution of the differential cross section. (No "simple theory" is possible to develop for the fluctuation in the integrated cross section analogous to the "simple theory" for fluctuation of the differential cross section). The largest discrepancy between ν_{exp} and ν_{theor} is seen at $\theta = 0°$, where the deviations generally are in the direction of large ν_{exp}. Possible contamination of unresolved alpha-particle energies from the reaction $O^{16}(C^{12},\alpha)Mg^{24}$ could increase ν_{exp}; the big discrepancy in the 2+, $\theta = 0°$ channel might result from such contamination. The aforementioned possible deviation from the randomness assumptions (6) (resulting from the possible existence of a direct amplitude) would also increase the fluctuation parameter ν_{exp}.

It has been postulated [8] that three "resonances" in the ground-state reaction are isolated resonances resulting from deformation of the compound system and clustering of the nucleons, which cause $\Gamma_i\Gamma_f$ to be very large. If such large resonances were isolated one would expect the same resonance to occur in the 2+ and 4+ excitation functions. No cross-correlated resonances between the 0+, 2+, and 4+ excitation functions are

seen at these "resonances", as is apparent from the example of fig. 2. This fact, in addition to the evidence from the present analysis that the reaction proceeds largely via a compound mechanism in the region of overlapping resonances, supports the idea that the three isolated "resonances" and the other peaks in the cross section can be interpreted as statistical fluctuations of the cross section.

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Appendix

A χ^2 distribution of ν degrees of freedom has the analytical form [27]

$$N_\nu(\chi^2) = \frac{1}{2\Gamma(\nu/2)}\left(\frac{\chi^2}{2}\right)^{\frac{\nu}{2}-1} \exp\left(-\frac{\chi^2}{2}\right), \tag{A 1}$$

which is normalized to unity by $\int_0^\infty N_\nu(\chi^2)d\chi^2 = 1$.

We take a set of n independent quantities, say σ_β, each with a decreasing exponential of probability distribution $P(\sigma_\beta) = \frac{1}{2}\exp\left(\frac{-\sigma_\beta}{2}\right)$ with mean value 2. Then a property of $\sigma = \sum_{\beta=1}^{n} \sigma_\beta$ relevant to our application is that it is distributed with a χ^2 distribution of $\nu = 2n$ degrees of freedom,

$$N_{2n}(\sigma) = \frac{1}{2\Gamma(n)}\left(\frac{\sigma}{2}\right)^{n-1}\exp\left(-\frac{\sigma}{2}\right). \qquad \text{(A 2)}$$

If $<>$ represents the average with respect to (A 2), then

$$\frac{<\sigma^2>}{<\sigma>^2} - 1 = \frac{\nu+2}{\nu} - 1 = \frac{2}{\nu} = \frac{1}{n}. \qquad \text{(A 3)}$$

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